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Rational Chebyshev Approximation on $[0, +\infty)$

J. A. ROULIER

North Carolina State University, Raleigh, North Carolina 27607

AND

G. D. TAYLOR*

Michigan State University, East Lansing, Michigan 48824

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I. INTRODUCTION

For any nonnegative integer m, let Π_m denote the collection of all real polynomials of degree at most m. For given $r > 0$ and $s > 1$, let $E(r, s)$ denote the unique ellipse in the complex plane with foci at $x = 0$ and $x = r$ and semi-major and semi-minor axes a and b, respectively, such that $b/a = (s^2 - 1)/(s^2 + 1)$. If $f(z)$ is any entire function, set

$$\tilde{M}_f(r, s) = \max\{|f(z)|: z \in E(r, s)\}.$$

In a recent paper [2], a Bernstein type of theory has been developed for the problem of approximating real valued functions on the half line $[0, \infty)$. Precisely, the following two results were proved.

THEOREM 1. *Let $f(x)$ be a real continuous function ($\neq 0$) on $[0, \infty)$, and assume that there exist a sequence of real polynomials $\{p_n(x)\}_{n=0}^{\infty}$, with $p_n \in \Pi_n$ for each $n \geqslant 0$, and a real number $q > 1$ such that*

$$\overline{\lim_{n\to\infty}} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\|_{L^\infty[0,\infty)} \right\}^{1/n} \leqslant \frac{1}{q} < 1. \tag{1}$$

Then, there exists an entire function $F(z)$ with $F(x) = f(x)$ for all $x \geqslant 0$, and F is of finite order ρ. In addition, for every $s > 1$, there exist constants $K = K(s, q) > 0$, $\theta = \theta(s, q) > 0$ and $r_0 = r_0(s, q) > 0$ such that

$$\tilde{M}_F(r, s) \leqslant K(\|f\|_{L^\infty[0,r]})^\theta \qquad \text{for all} \quad r \geqslant r_0. \tag{2}$$

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Also included in the theorem is a best possible upper estimate for ρ. It should also be noted that (1) implies that either f is identically a constant or $\lim_{x\to\infty} |f(x)| = \infty$.

THEOREM 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function with nonnegative coefficients and $a_0 > 0$. If there exist real numbers $s > 1$, $A > 0$, $\theta > 0$ and $r_0 > 0$ such that $\tilde{M}_f(r, s) \leqslant A(\|f\|_{L^\infty[0,r]})^\theta$ for all $r \geqslant r_0$ then there exists a sequence of real polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with $p_n \in \Pi_n$ for $n \geqslant 0$ such that*

$$\overline{\lim_{n\to\infty}} \left\{\left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\|_{L^\infty[0,\infty)}\right\}^{1/n} \leqslant s^{-[1/(1+\theta)]} < 1.$$

Thus, Theorem 1 states that geometric convergence on the half line of the reciprocals of polynomials to the reciprocal of a real continuous function implies that the function is the restriction of an entire function satisfying a growth condition (2) on certain ellipses. Furthermore, this growth condition implies that F is an entire function of finite order.

In the converse direction, if F is an entire function satisfying this growth condition and *having nonnegative Taylor coefficients* then there exists a sequence of real polynomials whose reciprocals converge geometrically to the reciprocal of F on the half line.

It is this additional assumption of nonnegative Taylor coefficients that motivated the work of this paper. This condition is not necessary for the conclusion of Theorem 2. For example, the function $F(z) = e^z + 2e^{-z}$ satisfies the other two hypotheses of Theorem 2 and the conclusion of Theorem 2 is valid for this function.

In this paper we shall present a new sufficient condition for geometric convergence to occur. It is our conjecture that Theorem 2 is true if one replaces the requirement on the Taylor coefficients of F with the assumption that F is either identically a constant or $\lim_{x\to\infty} |F(x)| = +\infty$. However, we have not succeeded in proving this and we would be pleased if an answer to this question could be found.

In the remainder of this paper we shall say that an entire function f has geometric convergence whenever there exist a sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$, $p_n \in \Pi_n$ for $n \geqslant 0$, and a real number $q > 1$, for which (1) holds. Also, we shall write $\|\cdot\|_{[0,r]}$ and $\|\cdot\|_{[0,\infty)}$ for $\|\cdot\|_{L^\infty[0,r]}$ and $\|\cdot\|_{L^\infty[0,\infty)}$, respectively.

II. MAIN RESULT

In this section we wish to prove a new theorem for geometric convergence to occur. This result will be a comparison type theorem. That is, we shall show that if f does not differ too much from a function known to have geometric convergence then f has geometric convergence. The proof that we shall give is self-contained; however, many of results that we shall use are special cases of some recent work by J. A. Roulier [3] and [4] on w-approximation. This work studies the problem of approximating continuous function on $[a, b]$ with polynomials in Π_m with respect to a weight function of the form

$$\omega(x) = \left(\prod_{i=1}^{n} | x - x_i |^{\alpha_i}\right)^{-1},$$

where $a \leqslant x_1 < x_2 < \cdots < x_n \leqslant b$ and α_i is a nonnegative real number for each $i = 1, \ldots, n$. Since the functions that we are approximating are always entire functions and the powers α_i are always integers here, we decided to simply develop the specific facts of w-approximation that we need within the proof without explicit reference to this more general study. We refer the reader to the papers referenced above for the details of w-approximation.

THEOREM 3. *Let f be an entire function having nonnegative zeros at precisely $\{x_i\}_{i=1}^{k}$, $0 \leqslant x_1 < x_2 < \cdots < x_k$, with respective orders $\beta_1, \ldots, \beta_k$ and assume there exist real numbers $K > 0$, $s_0 > 1$, $\theta > 0$ and $r_0 > 0$ such that*

$$\tilde{M}_f(r, s_0) \leqslant K(\|f\|_{[0,r]})^{\theta} \qquad \text{for all} \quad r \geqslant r_0 . \tag{3}$$

Further, assume there exist entire functions h and g such that

(i) *$f(z) = h(z) + g(z)$ for all $z \in C$,*

(ii) *$h(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \geqslant 0$ for $n = 0, 1, \ldots$, where h is not a polynomial and h has geometric convergence,*

(iii) *there exists $B > 0$ such that $g(x) \geqslant -B$ for all $x \geqslant 0$,*

(iv) *there exist $r_1 > 0$, $\psi > 0$ and $A > 0$, such that $g(x) \leqslant Ah^{\psi}(x)$ for all $x \geqslant r_1$,*

(v) *there exists a sequence of positive integers $\{n_j\}_{j=0}^{\infty}$ for which $0 \leqslant n_{j+1} - n_j \leqslant \rho$, ρ a fixed positive integer and*

$$g^{(n_j+1)}(x) \leqslant 0 \qquad \text{for all} \quad x \geqslant 0, \qquad j = 0, 1, \ldots . \tag{4}$$

Then there exist a sequence of real polynomials $\{s_n(x)\}_{n=0}^{\infty}$ with $s_n \in \Pi_n$ for each $n \geqslant 0$ such that

$$\varlimsup_{n\to\infty} \left\{\left\| \frac{1}{f(x)} - \frac{1}{s_n(x)} \right\|_{[0,\infty)}\right\}^{1/n} < 1.$$

Proof. Set

$$m = \beta_1 + \cdots + \beta_k, \qquad r_2 = \max(r_0, r_1, x_k + 1), \qquad w(x) = \prod_{i=1}^{k} (x - x_i)^{\beta_i}$$

and select $j_0 \geqslant 0$ such that $n_{j_0} \geqslant 3m$. Let $p_{3m}(x)$ and $q_{3m}(x)$ be the Hermite interpolating polynomials from Π_{3m} to $h(x)$ and $g(x)$, respectively, at the points $x_1, \ldots, x_k$ with respective orders $3\beta_1, \ldots, 3\beta_k$. Define h_1 and g_1 by

$$h_1(z) = \frac{h(z) - p_{3m}(z)}{w^3(z)}$$

and

$$g_1(z) = \frac{g(z) - q_{3m}(z)}{w^3(z)}.$$

Note that both h_1 and g_1 are entire functions. Select $r_3 \geqslant r_2$ such that $| z - x_i | \geqslant 1$ for $i = 1, \ldots, k$ and z on the boundary of $E(r_3, s_0)$ and

$$h(x) \geqslant \max \left(\sum_{i=0}^{3m} | b_i | x^i, \sum_{i=0}^{3m} | c_i | x^i \right)$$

for $x \geqslant r_3$, where $p_{3m}(x) = \sum_{i=0}^{3m} b_i x^i$ and $q_{3m}(x) = \sum_{i=0}^{3m} c_i x^i$. Thus, for $r \geqslant r_3$,

$$\begin{aligned} \tilde{M}_{h_1}(r, s_0) &= \max_{z \in \partial E(r, s_0)} \left| \frac{h(z) - p_{3m}(z)}{w^3(z)} \right| \\ &\leqslant \max_{z \in \partial E(r, s_0)} \left(| h(z)| + \sum_{i=0}^{3m} | b_i | \, | z |^i \right) \\ &\leqslant 2\tilde{M}_h(r, s_0) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \tilde{M}_{g_1}(r, s_0) &\leqslant \max_{z \in \partial E(r, s_0)} | g(z)| + \tilde{M}_h(r, s_0) \\ &\leqslant \tilde{M}_f(r, s_0) + 2\tilde{M}_h(r, s_0). \end{aligned} \tag{6}$$

For each $r \geqslant r_3$ and $j \geqslant j_0$, let $p_{n_j-3m}(x, r)$ be the best uniform approximation to h_1 from Π_{n_j-3m} on $[0, r]$,

$$\| h_1(x) - p_{n_j-3m}(x, r) \|_{[0,r]} = \inf_{s \in \Pi_{n_j-3m}} \| h_1(x) - s(x) \|_{[0,r]} = E^r_{n_j-3m}(h_1). \tag{7}$$

It is well known that p_{n_j-3m} is the Lagrange interpolating polynomial to h_1 on a certain set of points $0 < y_1 < \cdots < y_{n_j-3m+1} < r$. Set $p^*_{n_j}(x, r) = p_{n_j-3m}(x, r)\, w^3(x) + p_{3m}(x)$. Then, it is easily seen that $p^*_{n_j}$ is the Hermite interpolating polynomial to h on a certain set of points in $[0, r]$ including the set $x_1, ..., x_k$ with respective orders (at least) $3\beta_1, ..., 3\beta_k$ and

$$\| w^{-3}(x)(h(x) - p_{n_j}^*(x, r))\|_{[0,r]} = E_{n_j-3m}^r(h_1). \tag{8}$$

Similarly, let $q_{n_j-3m} \in \Pi_{n_j-3m}$ be the best uniform approximation to g_1 on $[0, r]$ with error $E_{n_j-3m}^r(g_1)$ and set $q_{n_j}^*(x, r) = q_{n_j-3m}(x, r)\, w^3(x) + q_{3m}(x)$. Then, $q_{n_j}^*(x, r)$ is the Hermite interpolating polynomial for g from Π_{n_j} on a certain set of nodes in $[0, r]$ and

$$\| w^{-3}(x)(g(x) - q_{n_j}^*(x, r))\|_{[0,r]} = E_{n_j-3m}^r(g_1). \tag{9}$$

Select $r_4 \geqslant r_3$ such that $h(r_4) \geqslant \max(2, 2B)$ and $w(x) \leqslant h(x)$ for all $x \geqslant r_4$ which is possible since h is not a polynomial. Write $f(z) = \hat{f}(z)\, w(z)$ where $\hat{f}$ is entire. Since $g(x) \geqslant -B$ for $x \geqslant 0$, we have that $f(x) \geqslant \frac{1}{2}h(x)$ for $x \geqslant r_4$. This, in turn, implies that $\hat{f}(x) \geqslant \frac{1}{2}[h(x)/w(x)]$ for $x \geqslant r_4$. Thus, there exists $\delta > 0$ such that $\hat{f}(x) \geqslant \delta$ for all $x \geqslant 0$. Now, fix $r \geqslant r_4$ and set $s_{n_j}^*(x, r) = p_{n_j}^*(x, r) + q_{n_j}^*(x, r) + E_{n_j}^r w^3(x)$, where $E_{n_j}^r = E_{n_j-3m}^r(h_1) + E_{n_j-3m}^r(g_1)$. As noted earlier, $p_{n_j}^*(x, r)$ and $q_{n_j}^*(x, r)$ are Hermite interpolating polynomials to h and g, respectively, on certain sets of nodes in $[0, r]$. Thus, $p_{n_j}^*(x, r)$ has all nonnegative coefficients since $h^{(j)}(x) \geqslant 0$ for $j = 0, 1, ...$ and $x \geqslant 0$. Also, the standard remainder formula for Hermite interpolation implies that $g_{n_j}^*(x, r) \geqslant g(x)$ for $x \geqslant r$ since $g^{(n_j+1)}(x) \leqslant 0$ for $x \geqslant 0$. Combining these facts and estimate (8), we have for $x \geqslant r$ and $j \geqslant j_0$,

$$\begin{aligned} S_{n_j}^*(x, r) &\geqslant p_{n_j}^*(x, r) + E_{n_j-3m}^r(h_1)\, w^3(x) + q_{n_j}^*(x, r) \\ &\geqslant h(r) + g(x) \\ &\geqslant \tfrac{1}{2}h(r) \end{aligned}$$

and

$$\left| \frac{1}{f(x)} - \frac{1}{s_{n_j}^*(x, r)} \right| \leqslant \frac{1}{|f(x)|} + \frac{1}{|s_{n_j}^*(x, r)|} \leqslant \frac{4}{h(r)}.$$

Since $g(x) \leqslant Ah^{\psi}(x)$ for $x \geqslant r$, we have that

$$\left| \frac{1}{f(x)} - \frac{1}{s_{n_j}^*(x, r)} \right| \leqslant K_1 \frac{1}{f^{\gamma}(r)} \tag{10}$$

for $r \geqslant r_4$ and $j \geqslant j_0$, where $\gamma = \{\max(1, \psi)\}^{-1}$ and $K_1 = 4(A + 1)$.

Due to the special form of $p_{n_j}^*(x, r)$ and $q_{n_j}^*(x, r)$ we may write $s_{n_j}^*(x, r) =$

$\hat{s}_{n_j-m}(x, r)\, w(x)$, where $\hat{s}_{n_j-m}(x, r) = Q_{n_j-m}(x, r) + E^r_{n_j} w^2(x)$. Now, for $0 \leqslant x \leqslant r$ and $j \geqslant j_0$,

$$|\hat{f}(x) - [Q_{n_j-m}(x, r)]| = |\, w^{-3}(x)\{f(x) - [p^*_{n_j}(x, r) + q^*_{n_j}(x, r)]\}|\, w^2(x)$$
$$\leqslant E^r_{n_j} w^2(x)$$

implying

$$\hat{s}_{n_j-m}(x, r) \geqslant \hat{f}(x)$$

and

$$\left| \frac{1}{f(x)} - \frac{1}{s^*_{n_j}(x)} \right| = |\, w^{-3}(x)[f(x) - s^*_{n_j}(x, r)]|\, \left| \frac{w(x)}{\hat{f}(x)\, \hat{s}_{n_j-m}(x, r)} \right|$$
$$\leqslant K_2 E^r_{n_j}, \tag{11}$$

where

$$K_2 = 2 \max_{0 \leqslant x} \left| \frac{w(x)}{\hat{f}^2(x)} \right|.$$

Using a result due to S. N. Bernstein [1, p. 91], we may estimate $E^r_{n_j}$ by

$$E^r_{n_j} \leqslant \frac{2}{(s_0 - 1)\, s_0^{n_j-3m}} [\tilde{M}_{h_1}(r, s_0) + \tilde{M}_{g_1}(r, s_0)] \tag{12}$$

since h_1 and g_1 are both entire functions. Using (5) and (6), we get

$$E^r_{n_j} \leqslant \frac{K_3}{s_0^{n_j}} [4\tilde{M}_h(r, s_0) + \tilde{M}_f(r, s_0)], \qquad \text{since} \quad r \geqslant r_3 \tag{13}$$

where $K_3 = 2\, s_0^{3m}/(s_0 - 1)$. Since h has geometric convergence there exist by Theorem 1 real numbers $K' > 0$, $\theta' > 0$ and $r' > 0$ such that $\tilde{M}_h(r, s_0) \leqslant K' \| h \|^{\theta'}_{[0,r]}$ for all $r \geqslant r'$. Without loss of generality we shall assume $r_4 \geqslant r'$. Combining this with (3), gives

$$E^r_{n_j} \leqslant \frac{K_3}{s_0^{n_j}} (4K' \| h \|^{\theta'}_{[0,r]} + K \| f \|^{\theta}_{[0,r]}). \tag{14}$$

From the inequality $f(x) \geqslant \frac{1}{2} h(x)$ for all $x \geqslant r_4$ and (i), we get that there exists a positive constant K_4, such that $\| f \|_{[0,r]} \geqslant K_4 \| h \|_{[0,r]}$ for all $r \geqslant r_4$. Hence, using this, (11), and (14)

$$\left| \frac{1}{f(x)} - \frac{1}{s^*_{n_j}(x, r)} \right| \leqslant \frac{K_5 \| f \|^{\varphi}_{[0,r]}}{s_0^{n_j}} \tag{15}$$

for $r \geqslant r_4$, $j \geqslant j_0$ and $0 \leqslant x \leqslant r$ where

$$\varphi = \max(\theta, \theta') \quad \text{and} \quad K_5 = K_2 \cdot K_3(4K'K_4^{-1} + K).$$

Now the fact that $\lim_{r\to\infty} f(r) = +\infty$ gives a positive integer $j_1 \geqslant j_0$ such that to each $j \geqslant j_1$ there corresponds an $r_j \geqslant r_4$ for which

$$\|f\|_{[0,r_j]} = f(r_j) = s_0^{n_j/(\varphi+\gamma)}.$$

Consequently, if we set $s_{n_j}^*(x) \equiv s_{n_j}^*(x, r_j)$ for each $j \geqslant j_1$ we see from (10) and (15) that

$$\left\| \frac{1}{f(x)} - \frac{1}{s_{n_j}^*(x)} \right\|_{[0,\infty)} \leqslant \frac{K_6}{s_1^{n_j}} \tag{16}$$

where $s_1 = s_0^{\gamma/(\gamma+\theta)}$ and $K_6 = \max(K_1, K_5)$. Finally, using the sequence of polynomials $s_n(x)$ where $s_n(x) \equiv s_{n_j}^*(x)$ for $n_j \leqslant n < n_{j+1}$ and $j \geqslant j_1$, gives

$$\overline{\lim_{n\to\infty}} \left\| \frac{1}{f(x)} - \frac{1}{s_n(x)} \right\|_{[0,\infty)}^{1/n} \leqslant \frac{1}{s_1} < 1$$

since $n_{j+1} - n_j \leqslant \rho$ for all j. ∎

We would like to remark that this Theorem remains true if one drops the requirement that h is not a polynomial. For in this case, it can be shown that the remaining hypotheses imply that f is also a polynomial.

Using this theorem, it readily follows that $f(x) = e^x + ce^{-x}$ has geometric convergence for each real constant c. We feel that an approach in this direction may prove that an entire function satisfying the growth condition and tending to $+\infty$ as $x \to \infty$ has geometric convergence. We had hoped to apply Theorem 3 to such a function by carefully separating its Taylor series into two parts. However, we have not succeeded and this remains open. We also feel that the hypotheses (iii)–(v) may be successfully weakened without affecting the truth of the theorem and this also is an open question. For example, Theorem 2 implies that $f(x) = e^x + \cos x$ has geometric convergence. We conjecture that $g(x) = e^x + e^4 \cos x$ has geometric convergence. However, Theorem 3 is not readily applied here as the obvious decomposition of g does not satisfy the hypotheses of Theorem 3.

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